Math 352 HW. # 3

Homework problems are taken from "Real Analysis" by N. L. Carothers. The problems are color coded to indicate level of difficulty. The color green indicates an elementary problem, which you should be able to solve effortlessly. Yellow means that the problem is somewhat harder. Red indicates that the problem is hard. You should attempt the hard problems especially.

Unless the contrary is explicitly stated, all numbers that are mentioned in these exercises are understood to be real.

1. If K is a nonempty compact subset of $\mathbb{R}$, show that sup K and inf K are elements of K.

2. Let $E = \{x \in Q : 2 < x^2 < 3\}$, considered as a subset of $\mathbb{Q}$ (with its usual metric.) Show that E is closed and bounded but *not* compact.

3. If A is compact in M and B is compact in N, show that A × B is compact in M × N.

4. If A is compact in M, prove that diam(A) is finite. Moreover, if A is nonempty, show that there exist points x and y in A such that diam(A) = d(x, y).

5. Given $c_n \geq 0$ for all n, prove that the set $\{x \in \ell_2 : \ |x| \leq c_n, \ \ n \geq 1\}$ is compact in ℓ_2 if and only if $\sum_{n=1}^{\infty} c_n^2 < \infty$.

6. Let E be a noncompact subset of $\mathbb{R}$. Find a continuous function $f : E \to \mathbb{R}$ that is (i) not bounded; (ii) bounded but has no maximum value.

7. Given an arbitrary metric space M, show that a decreasing sequence of nonempty *compact* sets in M has nonempty intersection.

8. Let F and K be disjoint, nonempty subsets of a metric space M with F closed and K compact. Show that $d(F, \ K) = \inf\{d(x, \ \ y) : x \in F, \ \ y \in K\} > 0$. Show that this may fail if we assume only that F and K are disjoint closed sets.

9. Show that any Lipschitz map *f*: (M, d) → (N, p) is uniformly continuous. In particular, any isometry is uniformly continuous.

10. Prove that a uniformly continuous map sends Cauchy sequences into Cauchy sequences.

11. If f: (0, 1) → ℝ is uniformly continuous, show that $\lim_{x\to 0^+} f(x)$ exists. Conclude that f is bounded on (0, 1).

12. Give an example of a bounded continuous map f: ℝ → ℝ that is not uniformly continuous. Can an unbounded continuous function f: ℝ → ℝ be uniformly continuous? Explain.

13. Prove that f: (M, d) → (N, p) is uniformly continuous if and only if $\rho(f(x_n),\ f(y_n)) \to 0$ for any pair of sequences $\{x_n\}$ and $\{y_n\}$ in M satisfying $d(x_n,\ y_n) \to 0$ [Hint: For the backward implication, assume that f is *not* uniformly continuous and work toward a contradiction.]

14. Given f: ℝ → ℝ and a ϵ ℝ, define F(x) = [f(x) – f(a)]/(x – a) for x ≠ a. Prove that f is differentiable at a if and only if F is uniformly continuous in some punctured neighborhood of a.

15. A function f: ℝ → ℝ is said to satisfy a *Lipschitz condition of order α*, where α > 0, if there is a constant K < ∞ such that $|f(x) - f(y)| \le K|x - y|^\alpha$ for all x, y. Prove that such a function is uniformly continuous.

16. Show that any function f: ℝ → ℝ having a bounded derivative is Lipschitz of order 1. [Hint: Use the mean value theorem.]

17. The Lipschitz condition is interesting only for $\alpha \le 1$; show that a function satisfying a Lipschitz condition of order α > 1 is *constant*.